SOLUTION OF PROBLEMS IN NUMBER FOUR.

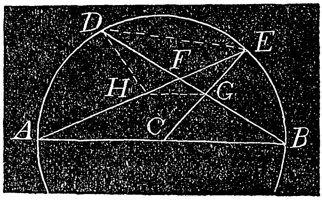
SOLUTIONS of problems in No. 4 have been received as follows:

From R. J. Adcock, 120 and 123; Marcus Baker, 119, 122 and 123; Dr. H. Eggers, 122; Edgar Frisby, 121; Henry Heaton, 119, 120, 121*, 122, 123 and 124; Prof. H. T. J. Ludwick, 121*; L. Regan, 119 and 122; E. B. Seitz, 119, 122, 123 and 124.

119. "If from one of the extremities, B, of the diameter, AB, of a given circle, any chord, BD, be drawn, and from the other extremity, A, and the centre C, two lines, AE, CE, be drawn to any point, E, in the circumference, cutting said, chord in the points F and G; then, $GF : GD :: (BF)^2 : (BA)^2$. Required the demonstraton."

DEMONSTRATION BY E. B. SEITZ, GREENVILLE, OHIO.

Draw GH parallel to BA; join DE and DH. The similar triangles DEF and HGF give $DF : HF :: EF : GF$, or $DF : EF :: HF : GF$; hence the triangles DFH and EFG are similar; therefore $\angle HDG = \angle GEH = \angle A$, and consequently the triangles ABF and DGH are similar.



The similar triangles ABF, HGF and DGH give $GF : GH :: BF : BA$, and $GH : GD :: BF : BA$. Multiplying these proportions together, we have $GF : GD :: BF^2 : BA^2$.

120. [As the solutions of this question, that have been submitted, are only forms from which the numerical value of x and y can be found by approximation, and as the answer which was sent with the question by Dr. Oliver, the proposer, cannot be verified by assigning particular values to x and y, and is therefore erroneous, it is not likely that a general solution of the question can be obtained.]

121 "In a spherical triangle are given the sum of each angle and the side opposite, to solve the triangle."

SOLUTION BY EDGAR FRISBY, ESQ., NAVAL OBSERVATORY, WASH., D.C.

We have

$$(1)\ \frac{\sin A - \sin a}{\sin A + \sin a} = \frac{\sin B - \sin b}{\sin B + \sin b} = \frac{\sin C - \sin c}{\sin C + \sin c} = \frac{\tan\frac{1}{2}(A-a)}{\tan\frac{1}{2}(A+a)} \&c. = x \text{ say.}$$

(2) $\tan\frac{1}{2}(A-a)=lx$, $\tan\frac{1}{2}(B-b)=mx$, $\tan\frac{1}{2}(C-c)=nx$ if $\tan\frac{1}{2}(A+a)=l$ &c.

$$(3)\left\{\begin{aligned}\sin A&=[\tan\tfrac{1}{2}(A+a)+\tan\tfrac{1}{2}(A-a)]\cos\tfrac{1}{2}(A+a)\cos\tfrac{1}{2}(A-a)\\ &=\frac{l(1+x)}{\sqrt{[(1+l^2)(1+l^2x^2)]}};\ \cos A=\frac{1+l^2x}{\sqrt{[(1+l^2)(1+l^2x^2)]}}\\ \sin a&=\frac{l(1-x)}{\sqrt{[(1+l^2)(1+l^2x^2)]}};\ \cos a=\frac{1+l^2x}{\sqrt{[(1+l^2)(1+l^2x^2)]}}\end{aligned}\right.$$

and similar expressions for B, b, C and c; substituting these values in Cagnoli's equation

(4) $\sin B\sin C-\sin b\sin c=\cos B\cos C\cos a+\cos b\cos c\cos A$

we have

$$(5)\quad \frac{mn[(1+x)^2-(1-x)^2]}{\sqrt{[(1+m^2)(1+n^2)(1+m^2x^2)(1+n^2x^2)]}}$$
$$=\frac{(1-m^2x)(1-n^2x)(1+l^2x)-(1+m^2x)(1+n^2x)(1-l^2x)}{\sqrt{[(1+l^2)(1+m^2)(1+n^2)(1+l^2x^2)(1+m^2x^2)(1+n^2x^2)]}},\ \text{or}$$

$$(6)\qquad 2mnx\sqrt{[(1+l^2)(1+l^2x^2)]}=1+(m^2n^2-l^2n^2-l^2m^2)x^2,$$

which, on clearing of radicals, becomes

$$1-2(l^2m^2+m^2n^2+n^2l^2+2l^2m^2n^2)x^2+(l^4m^4+m^4n^4+n^4l^4-2l^2m^2n^4-2l^2m^4n^2$$
$$-2l^4m^2n^2-4l^2m^2n^2)x^4=0;$$

dividing by x^4 and solving we have

$$(7)\quad x^{-2}=l^2m^2+m^2n^2+n^2l^2+2l^2m^2n^2\pm2lmn\sqrt{[(1+l^2)(1+m^2)(1+n^2)]},\quad\text{or}$$

$$(8)\ 1-x^{-2}=(1+l^2)(1+m^2)(1+n^2)-l^2(1+m^2)(1+n^2)-m^2(1+n^2)(1+l^2)$$
$$-n^2(1+l^2)(1+m^2)\pm2lmn\sqrt{[(1+l^2)(1+m^2)(1+n^2)]}$$
$$=(1+l^2)(1+m^2)(1+n^2)$$
$$\times\left(1-\frac{l^2}{1+l^2}-\frac{m^2}{1+m^2}-\frac{n^2}{1+n^2}\mp2\frac{l}{\sqrt{(1+l^2)}}\cdot\frac{m}{\sqrt{(1+m^2)}}\cdot\frac{n}{\sqrt{(1+n^2)}}\right)$$
$$=\frac{1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma\mp2\cos\alpha\cos\beta\cos\gamma}{\sin^2\alpha\sin^2\beta\sin^2\gamma}.$$

If $l=\pm\cot\alpha$, $m=\pm\cot\beta$, $n=\pm\cot\gamma$, or $A+a=180^\circ\mp2\alpha$, $B+b=180^\circ\mp2\beta$, $C+c=180^\circ\mp2\gamma$, from (2).

In (8) let $x=\cos\varphi$, which is always possible, for from (1) we can show that x always lies between $+1$ and -1, and equation (8) becomes

$$(9)\ \tan\varphi=\frac{2\sqrt{[\cos\frac{1}{2}(\alpha+\beta+\gamma).\cos\frac{1}{2}(\beta+\gamma-\alpha).\cos\frac{1}{2}(\alpha-\beta+\gamma).\cos\frac{1}{2}(\alpha+\beta-\gamma)]}}{\sin\alpha\sin\beta\sin\gamma}$$

For the upper sign, by comparing equation (6) with (7) we see that the positive sign only is admissable.

If now we put $\alpha+\beta+\gamma=2\omega$ this equation becomes

$$\tan\varphi=\frac{+2\sqrt{[\cos\omega\cos(\omega-\alpha)\cos(\omega-\beta)\cos(\omega-\gamma)]}}{\sin\alpha\sin\beta\sin\gamma}$$

which can easily be proved to be always possible. The other root will be

$$\tan\varphi = \frac{+2\surd[-\sin\omega\sin(\omega-\alpha)\sin(\omega-\beta)\sin(\omega-\gamma)]}{\sin\alpha\sin\beta\sin\gamma};$$

whether this value is real or not can be immediately inferred by inspection; it can be proved that α, β, γ, ω, $(\omega-\alpha)$, $(\omega-\beta)$ and $(\omega-\gamma)$ always lie between $\pm$ 90° when the sides and angles are each less than 180°.

Again from (2) $\tan\frac{1}{2}(A-a) = lx = \cot\alpha\cos\varphi$, &c.

We have then $A+a = 180° \pm 2\alpha$, $B+b = 180° \pm 2\beta$, $C+c = 180° \pm 2\gamma$, $\alpha+\beta+\gamma = 2\omega$;

$$\tan\varphi = \frac{+2\surd[\cos\omega\cos(\omega-\alpha)\cos(\omega-\beta)\cos(\omega-\gamma)]}{\sin\alpha\sin\beta\sin\gamma},$$

which is always possible, or

$$\tan\varphi = \frac{+2\surd[-\sin\omega\sin(\omega-\alpha)\sin(\omega-\beta)\sin(\omega-\gamma)]}{\sin\alpha\sin\beta\sin\gamma},$$

$\tan\frac{1}{2}(A-a) = \cos\varphi\cot\alpha$, $\tan\frac{1}{2}(B-b) = \cos\varphi\cot\beta$, $\tan\frac{1}{2}(C-c) = \cos\varphi\cot\gamma$. It will not make any difference whether we use $A + a = 180° + 2\alpha$ or $A + a = 180° - 2\alpha$ &c., for the sign of $\tan\varphi$ will correspondingly change; $\tan\varphi$ and $\cos\varphi$ must have the same sign.

121*.—"It has been cloudy during the last seven days; what is the probability that it will be cloudy to-morrow?"

SOLUTION BY PROF. H. T. J. LUDWICK, SALISBURY, N. C.

Let the probability that it will be cloudy on any particular day be denoted by x, then $1 - x$ is the probability that it will not be cloudy.

As x may vary from 0 to 1, and since it has been cloudy seven days

$\therefore$ $\int_0^1 x^7 dx$ = whole number of possible producing causes

and $\int_0^1 x^8 dx$ = " " " favorable causes.

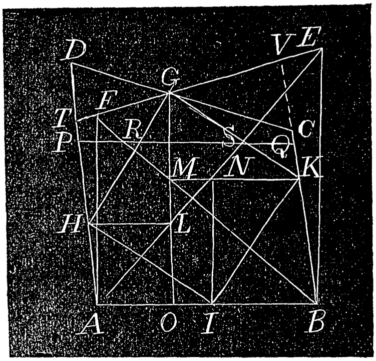
Therefore the probability that it will be cloudy on the 8th day

$$= \frac{\int_0^1 x^8 dx}{\int_0^1 x^7 dx} = \frac{8}{9}.$$

122. "To inscribe a square in a given quadrilateral."

SOLUTION BY HENRY HEATON, B. S., DES MOINES, IOWA.

Let $ABCD$ be the given quadrilateral. At any convenient distance from the base, as a, draw PQ parallel to the base and cutting AD in P and BC in Q, and lay off PS and QR each equal to a. Draw AS and prolong it to meet BE, drawn perpendicular to AB, in E. Draw BR and prolong it to meet AF, drawn perpendicular to AB, in F, and join EF cutting DC in G. From G draw GO perpendicular to AB cutting BF in M and AE in L, and draw MK and LH parallel to AB. Join GH and GK and draw KI and HI respectively parallel to GH and GK; then is $GHIK$ the inscribed square required.



Because LO in the triangle EAB equals GM in the triangle EFB, $\therefore$ $GL = MO$. By construction we have $BM : BR :: OM : a :: KM : a$; $\therefore$ $OM = KM$: Also, $AL : AS :: OL : a :: HL : a$; $\therefore$ $OL = HL$. Hence the right-angled triangles GMK and GLH are equal in all their parts and therefore HGK is a right angle and $GHIK$ is a square. Draw IN perpendicular to MK; then is $IN = GL = MO$. Hence I is on AB and $GHIK$ is the required square.

If the line EF should not intersect CD it is evident there can be no solution; if it should coincide with it, there will be an infinite number.

Problem No. 62 of the ANALYST may be constructed as follows:

Draw BT and AV in the given directions; BT cutting AD in T and AV cutting BC in V, and join TV cutting CD in G. Then is G the vertex of an angle of the required parallelogram.

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123. "Solve the equation $\sqrt[x]{x} = a$ and determine what values of a give real roots."

SOLUTION BY MARCUS BAKER, U. S. COAST SURVEY, WASHINGTON, D. C.

Taking the Napierian logarithms of both members we have

$$(\log x) \div x = \log a = b.$$

Put $x = 1 - y$ and substitute for $\log x$ its value from the logarithmic series,

$$\frac{y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \frac{1}{4}y^4 + \frac{1}{5}y^5 + \frac{1}{6}y^6 + \cdots}{1 - y} = -b = c,$$

or, performing the division,

$$y+\frac{3}{2!}y^2+\frac{11}{3!}y^3+\frac{50}{4!}y^4+\frac{274}{5!}y^5+\frac{1764}{6!}y^6+\frac{13068}{7!}y^7+\dots\dots=c.$$

Reverting the series we have

$$y=c-\frac{3}{2!}c^2+\frac{16}{3!}c^3-\frac{125}{4!}c^4+\frac{1296}{5!}c^5-\dots\dots$$

$$\therefore\ x=1-y=1-c+\frac{3}{2!}c^2-\frac{16}{3!}c^3+\frac{125}{4!}c^4-\frac{1296}{5!}c^5+\dots\dots$$

$$\text{or } x=1+\log a+\frac{3}{2!}(\log a)^2+\frac{4^2}{3!}(\log a)^3+\frac{5^3}{4!}(\log a)^4+\frac{6^4}{5!}(\log a)^5+.$$

If we now deduce the first differential coefficient of our expression $x^{\frac{1}{x}}=a$, a being a variable, and equate to zero we find $\log x=1$ or $x=e$, the Napierian base; hence x has its greatest value when

$$a=e^{\frac{1}{e}}=(2.71828)^{\frac{1}{2.71828}}=1.44467.$$

[Mr. Baker's answer to Query 2 was overlooked in making up our notices for No. 4. He says the equation can be solved, and for proof sends two solutions by approximation. Such solution, however, was probably not contemplated by the querist, as *all* eqations are thus solvable.]

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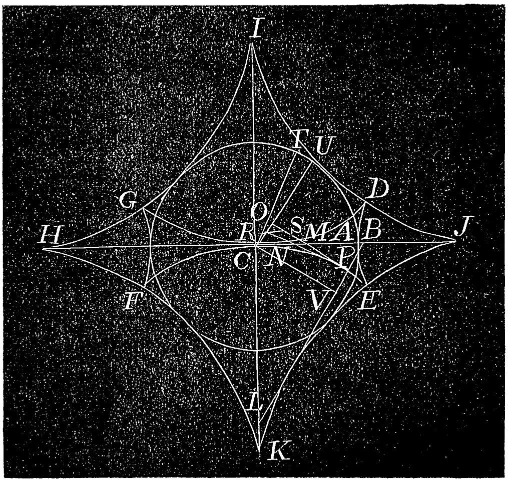
124. "A radius is drawn in the circle $x^2+y^2=a^2$ and from its extremity an ordinate. From the foot of the ordinate a line is drawn perpendicular to the radius. Find and discuss the envelope of this last line."

SOLUTION BY HENRY HEATON, B. S., DES MOINES, IOWA.

LET PO and PR be two consecutive positions of the given line. Put $CT=a$, $CN=x$, $TN=y$ and $MN=dx$. From the similar triangles CTN and CNR we find $CR=x^2\div a$; $\therefore\ RO=2xdx\div a$



Through P draw BL perpendicular and CV parallel to PR; then is $VL=2y^2\div a$ and $VB=2x^2\div a$; $\therefore\ BL=2a$.

BL, being perpendicular to PR, is normal to the required curve, and because the *envelope of the normal to any curve is the evolute of that curve*, KEJ, the envelope of BL, is the evolute of the required curve.

If $x_{\prime}$ and $y_{\prime}$ represent the coordinates of the point P, we shall have

$$x_{\prime} = x + \frac{xy^2}{a^2} \ldots . (1); \qquad y_{\prime} = -\frac{x^2 y}{a^2}; \ldots . (2)$$

By eliminating x and y from (1) and (2) we get

$$16a^2[x_{\prime} + \sqrt{(x_{\prime}^2 - 8y_{\prime}^2)}] = [3x_{\prime} + \sqrt{(x_{\prime}^2 - 8y_{\prime}^2)}]^3, \ldots (3)$$

which is the equation of the curve referred to rectangular coordinates.

From (1) and (2), $dx_{\prime} = a^{-2}(3y^2 - a^2)dx$ and $dy_{\prime} = a^{-2}(3y^2 - a^2)dy$. Therefore, if $z_{\prime}$ = the length of the required curve and z, that of the corresponding arc of the circle, $dz_{\prime} = a^{-2}(3y^2 - a^2)dz$. . . . . (4)

Put ρ = the radius of curvature of the envelope. Then, as the tangents of the circle and envelope, at their corresponding points, are parallel, it follows that $\rho : a :: dz_{\prime} : dz$. $\therefore \rho = a^{-1}(3y^2 - a^2)$. Hence, when $\rho = 0$, as at D, E, F, G, $y = a\sqrt{\frac{1}{3}}$, $y_{\prime} = \pm \frac{2}{9}a\sqrt{3}$, $x_{\prime} = \pm \frac{4}{9}a\sqrt{6}$, and the curve intersects its evolute in the points D, E, F, G, and consists of four branches, of which DAE is the involute of DJ and EJ, DCG, the involute of DI and GI, &c.

From (4) we have $dz_{\prime} = a^{-2}(3y^2 - a^2)dz = a^{-1}(3y^2 - a^2)(a^2 - y^2)^{-\frac{1}{2}}dy$; or, because the whole envelope is four times CD plus four times DA,

$$z_{\prime} = \frac{4}{a}\int_{a\sqrt{\frac{1}{3}}}^{0} \frac{(3y^2 - a^2)dy}{\sqrt{(a^2 - y^2)}} + \frac{4}{a}\int_{a\sqrt{\frac{1}{3}}}^{a} \frac{(3y^2 - a^2)dy}{\sqrt{(a^2 - y^2)}}$$

$$= 4a(\sqrt{2} - \sin^{-1}\sqrt{\tfrac{1}{3}}) + a\pi.$$

If A = the whole area of the envelope, we have

$$dA = y_{\prime}dx_{\prime} = a^{-4}y^2(3y - a^2)(a^2 - y^2)^{\frac{1}{2}}dy;$$

$$\therefore A = \frac{4}{a^4}\int_0^a (3y^2 - a^2)(a^2 - y^2)^{\frac{1}{2}}y^2dy = \tfrac{1}{8}a^2\pi.$$

The equation of the curve JEK, is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (2a)^{\frac{2}{3}}$. Its length is $3a$, $JE = a$, and $KE = 2a$. Being the envelope of BL it is easily described, and hence furnishes a convenient means of describing the required curve.

[Mr. Seitz has also given a very elegant solution of this problem. He employs polar coordinates and obtains for the equation of the curve

$$\rho = \frac{8a\sin\theta\{\sqrt{[(1-\sin\theta)(1+3\sin\theta)]} \pm \sqrt{[(1+\sin\theta)(1-3\sin\theta)]}\}}{\{\sqrt{[(1+\sin\theta)(1+3\sin\theta)]} \pm \sqrt{[(1-\sin\theta)(1-3\sin\theta)]}\}^2}$$

and for length and area, respectively,

$$s = 4\int_0^{\frac{2}{3}a\sqrt{3}} \frac{\rho d\rho}{(\rho^2 - p^2)^{\frac{1}{2}}} + 4\int_a^{\frac{2}{3}a\sqrt{3}} \frac{\rho d\rho}{(\rho^2 - p^2)^{\frac{1}{2}}} = a(4\sqrt{2} + \sin^{-1}\tfrac{1}{3}),$$

$$A = 2\int_a^{\frac{2}{3}a\sqrt{3}} \frac{p\rho d\rho}{(\rho^2 - p^2)^{\frac{1}{2}}} - 2\int_0^{\frac{2}{3}a\sqrt{3}} \frac{p\rho d\rho}{(\rho^2 - p^2)^{\frac{1}{2}}} = \tfrac{1}{8}\pi a^2.]$$